Quick Guide to Derivatives

For Ryan Holben's Math 2A class, Fall 2014 at UC Irvine[1].

Part I
Derivative Rules

1 Algebra with derivatives

Derivatives distribute over sums and differences:

$$\frac{d}{dx}\left[f(x)+g(x)\right]=\frac{d}{dx}f(x)+\frac{d}{dx}g(x)$$

and

$$\frac{d}{dx}\left[f(x)-g(x)\right]=\frac{d}{dx}f(x)-\frac{d}{dx}g(x)$$

Derivatives do **not** distribute over products of functions. See the **product rule** in the next section for that.

We can pull constant multiples out of derivatives:

$$\frac{d}{dx}\left[c\cdot f(x)\right]=c\cdot\frac{d}{dx}f(x)$$

2 Essential derivative rules

Derivative of a constant: $\frac{d}{dx}c=0$

Power rule: $\frac{d}{dx}x^n=n\,x^{n-1}$

Product rule: $\frac{d}{dx}f(x)\cdot g(x)=f^{'}(x)\cdot g(x)+f(x)\cdot g^{'}(x)$

Quotient rule: $\frac{d}{dx}\frac{f(x)}{g(x)}=\frac{f^{'}(x)\cdot g(x)-f(x)\cdot g^{'}(x)}{\left[g(x)\right]^2}$

Chain rule: $\frac{d}{dx}f\left(g\left(x\right)\right)=f^{'}\left(g\left(x\right)\right)\cdot g^{'}\left(x\right))$

[1] Last updated November 14th, 2014

3 Derivatives of specific functions

3.1 Exponentials and logarithms

$$\text{Exponential:} \qquad \frac{d}{dx}a^x = a^x \ln(a)$$

$$\hookrightarrow e: \qquad \frac{d}{dx}e^x = e^x$$

$$\text{Logarithms:} \qquad \frac{d}{dx}\log_a(x) = \frac{1}{x\ln(a)}$$

$$\hookrightarrow \text{Natural logarithm:} \qquad \frac{d}{dx}\ln(x) = \frac{1}{x}$$

$$\hookrightarrow \text{Natural logarithm of } |x|: \qquad \frac{d}{dx}\ln|x| = \frac{1}{x}$$

3.2 Trigonometric functions

$$\frac{d}{dx}\sin(x) = \cos(x) \qquad \frac{d}{dx}\cos(x) = -\sin(x) \qquad \frac{d}{dx}\tan(x) = \sec^2(x)$$

$$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x) \qquad \frac{d}{dx}\sec(x) = \sec(x)\tan(x) \qquad \frac{d}{dx}\cot(x) = -\csc^2(x)$$

3.3 Inverse trigonometric functions

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$$

For the sake of completeness, here are the remaining inverse trigonometric derivatives. We will not use them in this class, however.

$$\frac{d}{dx}\csc^{-1}(x) = \frac{-1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}\cot^{-1}(x) = \frac{-1}{1+x^2}$$

Part II
Techniques

4 Implicit differentiation

Example: Given the following implicit definition of a function, find y':

$$3x^2 + xy + y^4 - \cos(y) = 23$$

Solution: Take the derivative of both sides with respect to the variable x.

$$\frac{d}{dx}\left(3x^2 + xy + y^4 - \cos(y)\right) = \frac{d}{dx}(23)$$

Remember that **y is a function**. So the derivative of xy is a **product rule**, and the derivative of y^4 as well as the derivative of $\cos(y)$ are **chain rules**.

$$6x + \left(1 \cdot y + x \cdot y' + 4\,(y)^3 \cdot y'\right) - \left(-\sin(y) \cdot y'\right) = 0$$

$$6x + y + xy' + 4y^3y' + \sin(y)y' = 0$$

Now group all y' terms on the left side, and all other terms on the right.

$$xy' + 4y^3y' + \sin(y)y' = -6x - y$$

Factor out y' on the left and factor out the negative on the right.

$$y'\left(x + 4y^3 + \sin(y)\right) = -\,(6x + y)$$

Finally, divide so that we have isolated the derivative, y'.

$$y' = -\frac{6x + y}{x + 4y^3 + \sin(y)}$$

Notice that our answer involves both x and y. That is okay, since our original equation was not solved for y as a function of x.

5 Using logarithms to remove exponents

If you have a function with a function of x in both the base and the exponent, we can use a logarithm to bring the exponent down before taking the derivative.

Example: Find the derivative of

$$x^{\sin(x)}$$

Solution: First we make a full equation by writing

$$y = x^{\sin(x)}$$

We can't simply take a logarithm of $x^{\sin(x)}$ on its own and then take its derivative, because that will change our final answer. That is why we have made an equation first, so that we can take a logarithm of **both sides**. This way our final answer will still be correct.

Now take the **natural logarithm** of each side.

$$\ln\, y = \ln\, x^{\sin(x)}$$

Using the exponent rule for logarithms, we can bring down the exponent of $\sin(x)$.

$$\ln\, y = \sin(x)\ln\, x$$

Now take the **implicit derivative** of each side, with respect to x.

$$\tfrac{d}{dx}\ln\, y = \frac{d}{dx}\sin(x)\ln\, x$$

The derivative of $\ln\, y$ is $\frac{1}{y} \cdot y'$, by the chain rule. The right side of the equation uses the product rule.

$$\frac{y'}{y} = \cos(x) \cdot \ln\, x + \sin(x) \cdot \frac{1}{x}$$

Now multiply both sides by y to isolate y'.

$$y' = y\left(\cos(x) \cdot \ln\, x + \sin(x) \cdot \frac{1}{x}\right)$$

Unlike the previous example, here our derivative should involve just x, and not y. This is because the function that we are being asked to find the derivative of is **not** implicitly defined. So now simply plug in $y = x^{\sin(x)}$

$$y' = x^{\sin(x)} \left(\cos(x) \ln x + \sin(x) \frac{1}{x} \right)$$

6 Logarithmic differentiation

Logarithmic differentation is a technique in which we take the natural logarithm of a function before taking its derivative. We then use logarithm rules to avoid having to do the product, quotient, and even chain rule when taking the derivative.

The technique is a more general form of the previous example.

Example: Compute

$$\frac{d}{dx} \frac{\sqrt{x^2 - 3}}{x^7 \sin(x)}$$

Solution: A key thing to point out here is that logarithmic differentation is used to make taking derivatives easier. However, it's rarely required. Therefore, it is unlikely a problem will explicitly tell you to use this technique!

So how do we know when to use it? Well, in this example, I see a several terms multiplied or divided by each other. Additionally, I see several exponents (specifically $\frac{1}{2}$ and 7). All of these things that I observe will make for annoying derivatives, but can be **easily simplified if they were inside a logarithm**. Let's begin. The approach here is identical to the previous example's.

First make an equation.

$$y = \frac{\sqrt{x^2 - 3}}{x^7 \sin(x)}$$

Take a natural logarithm of each side.

$$\ln y = \ln \left(\frac{\sqrt{x^2 - 3}}{x^7 \sin(x)} \right)$$

Now **before we take any derivatives, apply logarithm rules**.

$$\ln y = \ln \left(\sqrt{x^2 - 3} \right) - \ln \left(x^7 \sin(x) \right)$$

$$\ln y = \ln \left(x^2 - 3 \right)^{\frac{1}{2}} - \left(\ln \left(x^7 \right) + \ln \sin(x) \right)$$

$$\ln y = \frac{1}{2} \ln \left(x^2 - 3 \right) - 7 \ln x - \ln \sin(x)$$

We have simplified our logarithms as far as we can. Now we are ready to take the derivative.

$$\frac{d}{dx} \ln y = \frac{d}{dx} \left[\frac{1}{2} \ln \left(x^2 - 3 \right) - 7 \ln x - \ln \sin(x) \right]$$

$$\frac{y'}{y} = \frac{1}{2} \frac{2x}{x^2 - 3} - 7 \frac{1}{x} - \frac{\cos(x)}{\sin(x)}$$

Multiply both sides by y and simplify a bit.

$$y' = y \left(\frac{x}{x^2 - 3} - \frac{7}{x} - \cot(x) \right)$$

Once again, since the problem was originally **not** an implicit differentiation problem, our final answer should be given entirely in terms of x.

$$y' = \frac{\sqrt{x^2 - 3}}{x^7 \sin(x)} \left(\frac{x}{x^2 - 3} - \frac{7}{x} - \cot(x) \right)$$